Advanced Math. Models & Applications
Vol.2, No.1, 2017, pp.1-5

ON A SOLUTION OF THE CONVOLUTION TYPE VOLTERRA EQUATION OF THE 1ST KIND

A.L. Karchevsky[1]

[1]Sobolev Institute of Mathematics of the Siberian Branch of RAS, Novosibirsk, Russia

Abstract. In this work theorems of existence and uniqueness of the solution of the convolution type Volterra integral equation of the 1st kind are proved. Necessary and sufficient conditions are formulated for the solution that belongs to class of decomposable in the Fourier series continuous functions whose coefficients tend to zero as $k^{-(1+\alpha)}$ $(\alpha > 0)$. Based on the proofs, a new method for numerical solution of this integral equation is proposed.

Keywords: *Volterra integral equation, convolution type equation, numerical solution, necessary and sufficient conditions.*

AMS Subject Classification: *45D05.*

Corresponding Author: *Prof. Andrey Karchevsky**,** Sobolev Institute of Mathematics of the Siberian Branch of RAS, 4 Acad.Koptyuq avenue, 630090, Novosibirsk, Russia,*
e-mail: karchevs@math.nsc.ru

Manuscript received: *20 December, 2016*

1. Introduction

In this work the theorems of existence and uniqueness of the solution of the convolution type Volterra integral equation of the 1st kind are proved. Necessary and sufficient conditions are formulated for the solution that belongs to class of decomposable in the Fourier series continuous functions whose coefficients tend to zero as $k^{-(1+\alpha)}$ $(\alpha > 0)$. Based on the proofs, a new method for numerical solution of this integral equation is proposed.

Usually the problem of existence of the solution of the Volterra integral equation of the 1st kind is solved with the help of the converting integral equation of the 1st kind to the integral equation of the 2nd kind. The existence and uniqueness of the solution of the Volterra integral equation of the 2nd kind provides the existence and uniqueness of the solution of the Volterra integral equation of the 1st kind. If the conversion to the integral equation of the 2nd kind is impossible, then the question of existence of the solution remains open. In some special cases, there are formulas for the solution of the equation of the 1st kind. For example, the case of the degenerate kernel (kernel *K(x, t)* can be represented in the form $a_1(x)b_1(t) + \cdots + a_n(x)b_n(t)$, the Abel equation, kernel with logarithmic singularity, etc. These and other examples can be found here [1, p. 28-44]. In the general case for the solution of the convolution type Volterra integral equation of the 1st kind the Titchmarsh theorem is known [5]: if the solution exists and the kernel satisfies the known condition, then the solution is unique.

A fair complete the information about theory and practice of solving integral equations of various types can be found in the handbooks [1, 3] and in the Internet page [4].

2. Main result

Consider the convolution type Volterra integral equation of the 1st kind

$$\int_0^t K(t-s)f(s)ds = g(t) \tag{1}$$

It is assumed that the right side of $g(t)$ satisfies the condition $g(0) = 0$.

We do not assume that any of the conditions, allowing one to reduce the integral equation (1) to the integral Volterra equation of 2nd kind, exist. On the contrary, the case does not allow one to do it (for example, $K^{(n)}(0) = 0$ $(n = 0,1,2,\dots)$), is possible.

Consider the space $F^\beta[0,T]$ of continuous functions $f(t)$ on the interval $[0,T]$, which can be expanded in a Fourier series

$$f(t) = b_0 + \sum_{k=1}^{\infty}\big(a_k sin(\omega_k t) + b_k cos(\omega_k t)\big), \qquad \omega_k = 2\pi k/T\,, \tag{2}$$

here $|a_k| \le Ak^{-\beta}$ and $|b_k| \le Bk^{-\beta}$ (A and B are some nonnegative constants).
Let O_0^ε be a punctured ε-neighborhood of the point $t = 0$.

Theorem 1. Let function $K(t) \not\equiv 0$ be a piecewise continuous function on the interval $[0,T]$ and $|K(t)| \le m_0$. For the existence of the solution $f(t) \in F^{1+\alpha}[0,T]$ $(\alpha > 0)$ of the integral equation (1) it is necessary and sufficient if the function $g(t)$ on the interval $[0,T]$ can be represented in the form

$$g(t) = b_0 g_0(t) + \sum_{k=1}^{\infty}\big(a_k g_k(t) + b_k g'_k(t)\big), \tag{3}$$

where
1. function $g_0(t)$ has the form

$$g_0(t) = \int_0^t K(s)ds\,, \tag{4}$$

2. functions $g_k(t)$ $(k = 1,2,\dots)$ solve the problems

$$g''_k(t) + \omega_k^2(t) = K(t)\,, \quad g_k(0) = 0\,, \quad g'_k(0) = 0\,; \tag{5}$$

3. $|a_k| \le Ak^{-\alpha}$ and $|b_k| \le Bk^{-(1+\alpha)}$ and $A > 0, B > 0$ and b_0 are some constants.

The solution has the form

$$f(t) = b_0 + \sum_{k=1}^{\infty}\left(\frac{a_k}{\omega_k} sin(\omega_k t) + b_k cos(\omega_k t)\right). \tag{6}$$

Proof. *Necessity.* Let $f(t) \in F^{1+\alpha}[0,T]$ be the solution of the integral equation (1) where $\alpha > 0$. Then due to the uniform convergence of the series (2) the function $g(t)$ can be represented in the form

$$g(t) = \tilde{b}_0 g_0(t) +$$

$$+\sum_{k=1}^{\infty}\left(\tilde{a}_k\int_0^t K(s)sin\big(\omega_k(t-s)\big)ds+\tilde{b}_k\int_0^t K(s)cos\big(\omega_k(t-s)\big)ds\right).$$

It easy to see that the function

$$g_k(t)=\frac{1}{\omega_k}\int_0^t K(s)sin\big(\omega_k(t-s)\big)ds \tag{7}$$

is a solution of the problem (5) and $g_k(t)\in C^1[0,T]$.

Thus the function $g(t)$ has the form (3) and $b_0=\tilde{b}_0$, $a_k=\tilde{a}_k\omega_k$ and $b_k=\tilde{b}_k$ $(k=1,2,\dots)$. Due to $|K(t)|\le m_0$, this series converges uniformly, hence it converges to the continuous function. Due to boundary conditions (5) $g(0)=0$.

Sufficiency. Let function $g(t)$ have the form (3). Solution of the problem (5) can be represented in the form (7). It easy to see that the equality

$$\int_0^t K(s)\left[b_0+\sum_{k=1}^{\infty}\left(\frac{a_k}{\omega_k}sin\big(\omega_k(t-s)\big)+b_kcos\big(\omega_k(t-s)\big)\right)\right]ds=g(t)$$

holds. If $|a_k/\omega_k|\le 2\pi T^{-1}Ak^{-(1+\alpha)}$ and $|b_k|\le Bk^{-(1+\alpha)}$ then the series in the square brackets converges uniformly, hence it converges to the continuous function $f(t)$ and $f(t)\in \mathrm{F}^{1+\alpha}[0,T]$.

Theorem is proved.

If there is such $\varepsilon>0$ that $K(t)\ne 0$ for $t\in\mathcal{O}_0^{\varepsilon}$, then the uniqueness of the solution of the integral equation (1) follows from the Titchmarsh theorem [5] (see also [6, p.49]).

Theorem 2. Let the functions $g^1(t)$ and $g^2(t)$ can be represented in the form

$$g^i(t)=b_0^i g_0(t)+\sum_{k=1}^{\infty}\left(a_k^i g_k(t)+b_k^i g_k'(t)\right),\quad i=1,2, \tag{8}$$

where b_0^i , a_k^i and b_k^i $\left(k=\overline{1,k}\,, i=1,2\right)$ satisfy the condition 3 from Theorem 1. If $K(t)\not\equiv 0$

$$|b_0^1-b_0^2|\le\delta,\quad |a_k^1-a_k^2|\le\delta k^{-\alpha}\,,\quad |b_k^1-b_k^2|\le\delta k^{-(1+\alpha)}\,,\quad k=1,2,\dots \tag{9}$$

then for the solutions $f^1(t)$ and $f^2(t)$ of the integral equation (1) the inequality

$$|f^1(t)-f^2(t)|\le C\delta \tag{10}$$

holds, where $C=C(T,\alpha)$ is constant.

Proof. First of all, it is clear that the difference $g(t)=g^1(t)-g^2(t)$ can be represented as (3), therefore, it corresponds to the solution $f(t)=f^1(t)-f^2(t)$ of the equation (1), which is represented in the form (6).

Rating the difference

$$|f^1(t)-f^2(t)|=\left|b_0^1-b_0^2+\sum_{k=1}^{\infty}\left(\frac{a_k^1-a_k^2}{\omega_k}\sin(\omega_k t)+(b_k^1-b_k^2)cos(\omega_k t)\right)\right|\le$$

$$\le\left[1+(T/2\pi+1)\sum_{k=1}^{\infty}\frac{1}{k^{1+\alpha}}\right]\delta=[1+(T/2\pi+1)\zeta(1+\alpha)]\delta\equiv C(T,\alpha)\delta\,,$$

Where $\zeta(\bullet)$ is the Riemann ζ-function, we get the proof of the theorem.

Theorem is proved.

Note that from Theorem 2 it follows the uniqueness and stability of the solution of the integral equation (1) in $F^{\beta}[0,T]$ without the assumption that $K(t) \neq 0$ for $t \in Q_0^{\xi}$.

3. Numerical method for solving the integral equation

The proved theorem enables us to offer a numerical method for solving the convolution type Volterra integral equation of the 1st kind, which is not found in the reference literature [1-4]. Let solution of the integral equation (1) should be found on the interval $[0,T]$. As a rule, the right side of the equation is known with some error, i.e., $g_\delta(t)$ is given.

Let $g_0(t)$ and $g_k(t)$ be calculated $(k = \overline{1,N})$ (see (4)-(5)). We need to solve the following minimization problem:

$$J[a_1,\dots,a_N,b_0,b_1,\dots,b_N] \equiv \int_0^T n^2(t)dt \to min, \qquad (11)$$

where

$$n(t) = b_0 g_0(t) + \sum_{k=1}^{N} \left(a_k g_k(t) + b_k g_k'(t)\right) - g_\delta(t) \,.$$

The functional (11) is quadratic and the conjugate gradient method can be used [7] to its minimization.

The efficiency of the proposed method can be shown an example, where

$$K(t) = e^{-1/t}\,,\;\; f(t) = 3\cdot 10^{-5} t(T-t)(T/2-t) + 2sin(t\,(T-t)/100) +$$
$$+9\left(1-cos\left(t(T-t)\right)/1000\right),\quad T = 200(s)\,,$$

and function $g(t)$ is calculated with the help of the calculation of the integral (1), after that random error is added to it: $g_\delta(t) = g(t) + \xi(t)\frac{P}{100}$, where ξ is a random value on $[-1,1]$ and P is a percentage of the error. The function $f(t)$ is chosen in this form because it has a smooth trend component and a small oscillations along its entire length. The result of solving is shown in the figure.

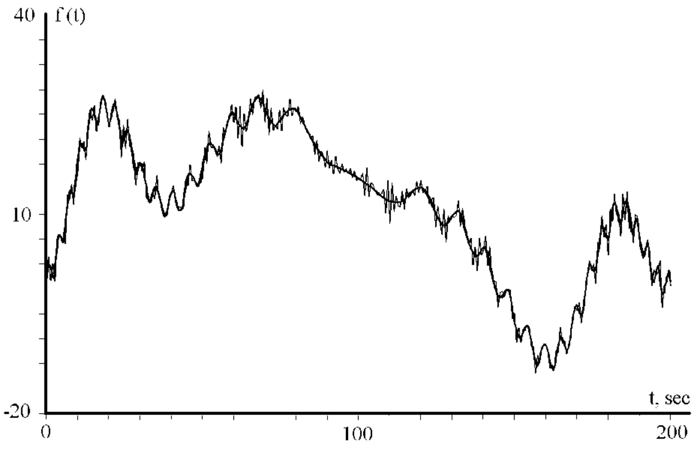

Figure. Result of solving the integral equation (1).
Smooth line is exact solution, broken line is the result of the numerical solution, $P = 10\%$.

4. Acknowledgment

This work was supported by the Program of Presidium of RAS (0314-2015-0011) and the RFBR grant (15-01-09230).

References

1. Manzhirov A.V., Polyanin A.D., (2000) Integral Equations, Methods of Solution, Handbook, Moscow, Faktorial Press, 384 p. (in Russian).
2. Polyanin A.D., Manzhirov A.V., (2003) Handbook for Integral Equations, Moscow, Fizmatlit, 608 p. (in Russian).
3. Verlan A.F., Sizikov V.S., (1986) Integral Equations: Methods, Algorithms, Codes. Handbook, Kiev, Naukova Dumka, 543 p. (In Russian).
4. EqWorld, http://eqworld.ipmnet.ru
5. Titchmarsh E.C., (1939) The Theory of Functions, Oxford University Press, 460 p.
6. Bukhgeim A.L., (1983) Volterra Equations and Inverse Problems, Novosibirsk, Nauka, 207 p. (in Russian).
7. Vasil’ev F.P., (1988) Numerical Methods for Solving Extreme Problems, Moscow, Nauka, 552 p. (in Russian).